ON THE LENGTH OF CRITICAL ORBITS
OF STABLE QUADRATIC POLYNOMIALS

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Abstract. We use the Weil bound of multiplicative character sums, together with some recent results of N. Boston and R. Jones, to show that the critical orbit of quadratic polynomials over a finite field of \(q\) elements is of length \(O\left(q^{3/4}\right)\), improving upon the trivial bound \(q\).

1. Introduction

Let \(\mathbb{F}_q\) be a finite field of \(q\) elements. For a polynomial \(f \in \mathbb{F}_q[X]\) we define the sequence of iterations:

\[
    f^{(0)}(X) = X, \quad f^{(n)}(X) = f\left(f^{(n-1)}(X)\right), \quad n = 1, 2, \ldots
\]

Following \[1, 2, 8, 9\], we say that \(f\) is stable if all polynomials \(f^{(n)}\) are irreducible over \(\mathbb{F}_q\).

We now assume that \(q\) is odd.

As in \[9\], for a quadratic polynomial \(f(X) = aX^2 + bX + c \in \mathbb{F}_q[X], \ a \neq 0\), we define \(\gamma = -b/(2a)\) as the unique critical point of \(f\) (that is, the zero of the derivative \(f'\)) and consider the set

\[
    \text{Orb}(f) = \{f^{(n)}(\gamma) : n = 2, 3, \ldots\},
\]

which is called the critical orbit of \(f\). Clearly there is some \(t\) such that \(f^{(t)}(\gamma) = f^{(s)}(\gamma)\) for some positive integer \(s < t\). Then \(f^{(n+t)}(\gamma) = f^{(n+s)}(\gamma)\) for any \(n \geq 0\). Accordingly, for the smallest value of \(t_f\) with the above condition, we have

\[
    \text{Orb}(f) = \{f^{(n)}(\gamma) : n = 2, \ldots, t_f\}
\]

and \(#\text{Orb}(f) = t_f - 1\) or \(#\text{Orb}(f) = t_f - 2\) (depending on whether \(s = 1\) or \(s \geq 2\) in the above). It is shown in \[7, 8, 9\] that critical orbits play a very important role in the dynamics of polynomial iterations.

Trivially we have \(t_f \leq q + 1\). In fact, by the Birthday Paradox one expects that \(t_f\) is of order \(q^{1/2}\) (for a sufficiently large \(q\)). Indeed, it is natural to expect that the map \(x \mapsto f(x)\) behaves like a random map on \(\mathbb{F}_q\), for which the trajectory length is of this order; see \[4\] for a detailed treatment of cycle structure of random maps on finite sets. For example, the Pollard integer factorisation algorithm (where a quadratic
polynomial \( f(X) = X^2 + c \) is iterated in a residue ring; see [3, Section 5.2.1]) is based on this assumption.

Here we obtain a nontrivial upper bound on the orbit length of stable quadratic polynomials:

**Theorem 1.** For any odd \( q \) and any stable quadratic polynomial \( f \in \mathbb{F}_q[X] \) we have

\[
t_f = O \left( \frac{q^3}{4} \right).
\]

By [3, Proposition 3], a quadratic polynomial \( f \in \mathbb{F}_q[X] \) is stable if the adjusted orbit

\[
\overline{\text{Orb}}(f) = \{-f(\gamma)\} \cup \text{Orb}(f)
\]

contains no squares. We also recall that \( \alpha \in \mathbb{F}_q \) is a square if either \( \alpha = 0 \) or \( \alpha^{(q-1)/2} = 1 \), which can be tested (via repeated squaring) in \( O(\log q) \) field operations. Combining these with the bound of Theorem 1, we immediately obtain:

**Corollary 2.** For any odd \( q \), a quadratic polynomial \( f \in \mathbb{F}_q[X] \) can be tested for stability in time \( q^{3/4+o(1)} \).

Our proof is based on the Weil bound for multiplicative character sums with polynomials; see [6, Theorem 11.23].

Finally, we remark that estimating the size of the set of stable quadratic polynomials \( aX^2 + bX + c \in \mathbb{F}_q[X] \) is a very interesting question to which we hope our technique can apply as well.

2. **Proof of Theorem 1**

Let \( \chi \) be the quadratic character of \( \mathbb{F}_q \).

By [3, Proposition 3], if a quadratic polynomial \( f \in \mathbb{F}_q[X] \) is stable, then \( \text{Orb}(f) \) contains no squares, that is, \( \chi \left( f^{(n)}(\gamma) \right) = -1 \), \( n = 2, 3, \ldots \).

We now fix an integer parameter \( K \) and note that for any \( n \geq 1 \), we have simultaneously

\[
\chi \left( f^{(k+n)}(\gamma) \right) = -1, \quad k = 1, \ldots, K,
\]

which we rewrite as

\[
\chi \left( f^{(k)} \left( f^{(n)}(\gamma) \right) \right) = -1, \quad k = 1, \ldots, K.
\]

Since by the definition of \( t_f \) the values \( f^{(n)}(\gamma), n = 1, \ldots, t_f - 1 \), are pairwise distinct elements of \( \mathbb{F}_q \), we derive from (1) that

\[
t_f - 1 \leq \#T_q(K),
\]

where

\[
T_q(K) = \left\{ x \in \mathbb{F}_q : \chi \left( f^{(k)}(x) \right) = -1, k = 1, \ldots, K \right\}.
\]

We have

\[
\#T_q(K) = \frac{1}{2^K} \sum_{x \in \mathbb{F}_q} \prod_{k=1}^{K} \left( 1 - \chi \left( f^{(k)}(x) \right) \right)
\]

since for every \( x \in T_q(K) \) the product on the right hand side of (3) is \( 2^K \); otherwise it is 0 when \( \chi(f^{(k)}(x)) = 1 \) for at least one \( k = 1, \ldots, K \) (note that since by our assumption \( f^{(k)}(X) \) is irreducible over \( \mathbb{F}_q \), we have \( f^{(k)}(x) \neq 0 \) for \( x \in \mathbb{F}_q \)).
Just expanding the product in (3), we obtain \(2^k - 1\) character sums of the shape

\[
(-1)^\nu \sum_{x \in \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} f^{(k_j)}(x) \right), \quad 1 \leq k_1 < \ldots < k_\nu \leq K,
\]

with \(\nu \geq 1\) and one trivial sum that equals \(q\) (corresponding to the terms 1 in the product in (4)).

Clearly \(f^{(k)}(X)\) is a polynomial of degree \(2^k\). Furthermore, by our assumption, each polynomial \(f^{(k)}(X)\) is irreducible; therefore none of the polynomials

\[
\prod_{j=1}^{\nu} f^{(k_j)}(X) \in \mathbb{F}_q[X], \quad 1 \leq k_1 < \ldots < k_\nu \leq K,
\]

are a perfect square in the algebraic closure of \(\mathbb{F}_q\). Therefore the Weil bound (see [6, Theorem 11.23]) applies to every sum (4) and implies that each of them is \(O(2^K q^{1/2})\). Therefore

\[
\#T_q(K) = \frac{1}{2^K} q + O(2^K q^{1/2}).
\]

Choosing \(K\) to satisfy

\[
2^K \leq \frac{q}{4} < 2^{K+1}
\]

and combining (2) and (5), we conclude the proof.

### 3. Comments

It is certainly interesting to obtain nontrivial estimates on the size \(S_q\) of the set of triples \((a, b, c)\in\mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q\) which correspond to stable quadratic polynomials \(f(X) = aX^2 + bX + c\). Denoting by \(F_k(a, b, c)\) the \(k\)th element of the critical orbit of \(f\), we see that for any integer parameter \(K\) we have

\[
S_q \leq \#W_q(K),
\]

where

\[
W_q(K) = \{(a, b, c) \in \mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q : \chi(F_k(a, b, c)) = -1, \ k = 1, \ldots, K\},
\]

and as before \(\chi\) denotes the quadratic character of \(\mathbb{F}_q\). As in the proof of Theorem 1 we have

\[
\#W_q(K) \leq \frac{1}{2^K} \sum_{(a,b,c)\in\mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q} \prod_{k=1}^{K} (1 - \chi(F_k(a, b, c)))
\]

since for every triple \((a, b, c)\in\mathbb{W}_q(K)\) the product on the right hand side of (7) is \(2^K\); otherwise it is either 0 (when \(\chi(F_k(a, b, c)) = 1\) for at least one \(k = 1, \ldots, K\)) or 1 (when \(F_1(a, b, c) = \ldots = F_K(a, b, c) = 0\)).

Clearly \(F_k(a, b, c)\) is a rational function in \(a, b, c\) of degree at most \(O(2^k)\). Thus expanding the product in (7), we obtain \(2^K - 1\) character sums of the shape

\[
(-1)^\nu \sum_{(a,b,c)\in\mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} F_{k_j}(a, b, c) \right), \quad 1 \leq k_1 < \ldots < k_\nu \leq K,
\]

with \(\nu \geq 1\) and one trivial sum corresponding to 1 in (7). Assuming that one can prove that the Weil-type bound \(O(2^K q^{5/2})\) applies to all of them, we obtain from (6) that \(S_q = O(q^{3/2} + 2^K q^{5/2})\) and optimising the choice of \(K\) we derive...
$S_q = O(q^{11/4})$. In fact, for a nontrivial estimate of $S_q$ it is enough to show that almost all sums (8) admit a nontrivial estimate, which has actually been recently done in [5], where the bound $S_q = O(q^{14/5})$ is obtained.

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